

Vector bundles and theta functions on curves of genus 2 and 3

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Introduction

Let C be a smooth projective curve, of genus $g \geq 2$. The moduli space $\mathcal{SU}_C(r)$ of semi-stable vector bundles of rank r on C , with trivial determinant, is a normal projective variety, which can be considered as a non-abelian analogue of the Jacobian variety J_C . It is actually related to J_C by the following construction, which goes back (at least) to [N-R]. Let J^{g-1} be the translate of J_C parameterizing line bundles of degree $g-1$ on C , and $\Theta \subset J^{g-1}$ the canonical theta divisor. For $E \in \mathcal{SU}_C(r)$, consider the locus

$$\Theta_E := \{L \in J^{g-1} \mid H^0(C, E \otimes L) \neq 0\}.$$

Then either $\Theta_E = J^{g-1}$, or Θ_E is in a natural way a divisor in J^{g-1} , belonging to the linear system $|r\Theta|$. In this way we get a rational map

$$\theta : \mathcal{SU}_C(r) \dashrightarrow |r\Theta|$$

which is the most obvious rational map of $\mathcal{SU}_C(r)$ in a projective space: it can be identified to the map $\varphi_{\mathcal{L}} : \mathcal{SU}_C(r) \dashrightarrow \mathbb{P}(H^0(\mathcal{SU}_C(r), \mathcal{L})^*)$ given by the global sections of the determinant bundle \mathcal{L} , the positive generator of the Picard group of $\mathcal{SU}_C(r)$ [B-N-R].

For $r = 2$ the map θ is an embedding if C is not hyperelliptic [vG-I]. We consider in this paper the higher rank case, where very little is known. The first part is devoted to the case $g = 2$. There a curious numerical coincidence occurs, namely

$$\dim \mathcal{SU}_C(r) = \dim |r\Theta| = r^2 - 1.$$

For $r = 2$ θ is an isomorphism [N-R]; for $r = 3$ it is a double covering, ramified along a sextic hypersurface which is the dual of the ‘‘Coble cubic’’ [O]. We will prove:

Theorem A.— *For a curve C of genus 2, the map $\theta : \mathcal{SU}_C(r) \dashrightarrow |r\Theta|$ is generically finite (or, equivalently, dominant). It admits some fibers of dimension $\geq \lfloor \frac{r}{2} \rfloor - 1$.*

Our method is to consider the fibre of θ over a reducible element of $|r\Theta|$ of the form $\Theta + \Delta$, where Δ is general in $|(r-1)\Theta|$. The main point is to show that this fibre restricted to the stable locus of $\mathcal{SU}_C(r)$ is finite. The other elements of the fibre are the classes of the bundles $\mathcal{O}_C \oplus F$, with $\Theta_F = \Delta$; reasoning by induction on r we may assume that there are finitely many such F , and this gives the first assertion

of the theorem (§ 1). The second one follows from considering the restriction of θ to a particular class of vector bundles, namely the symplectic bundles (§ 2).

The method is not, in principle, restricted to genus 2 curves – but the geometry in higher genus becomes much more intricate. In the second part of the paper (§ 3) we will apply it to rank 3 bundles in genus 3. Our result is:

Theorem B.— *Let C be a curve of genus 3. The map $\theta : \mathcal{SU}_C(3) \rightarrow |3\Theta|$ is a finite morphism.*

This means that a semi-stable vector bundle of rank 3 on C has always a theta divisor; or alternatively (see e.g. [B1]), that the linear system $|\mathcal{L}|$ on $\mathcal{SU}_C(3)$ is base point free.

This is not a big surprise since the result is already known for a *generic* curve of genus 3 [R]. We believe, however, that the method is more interesting than the result itself. In fact we translate the problem into an elementary question of projective geometry: what are the continuous families of planes in \mathbb{P}^5 such that any two planes of the family intersect? It turns out that this question has been completely (and beautifully) solved by Morin [M]. Translating back his result into the language of vector bundles we get a complete list of the stable rank 3 bundles E of degree 0 such that $\Theta_E \supset \Theta$ (Theorem 3.1 below). Theorem B follows as a corollary.

I am very much indebted to C. Ciliberto for pointing out the paper of Morin and for making it accessible to me.

Notations :

Throughout the paper we will work with a complex curve C (smooth, projective, connected), of genus g . If E is a vector bundle on C , we will write $H^0(E)$ for $H^0(C, E)$, and $h^0(E)$ for its dimension.

1. Genus 2: the generic finiteness

In this section we assume $g = 2$. The first part of theorem A follows from a slightly more precise result:

Proposition 1.1.— *Let Δ be a general divisor in $|(r-1)\Theta|$. The fibre $\theta^{-1}(\Theta + \Delta)$ is finite and non-empty.*

(1.2) We will prove the Proposition by induction on r . Let $[E_0] \in \theta^{-1}(\Theta + \Delta)$. If it is not stable, it is the class of a direct sum $\bigoplus_i E_i$, so that $\Theta_{E_0} = \sum_i \Theta_{E_i}$; thus $[E_0]$ is the class in $\mathcal{SU}_C(r)$ of $\mathcal{O}_C \oplus F$ for some $F \in \mathcal{SU}_C(r-1)$ with $\Theta_F = \Delta$. By the induction hypothesis there exists only finitely many such F , and there exists at least one.

Thus we can assume that E_0 is stable. Let $E := E_0^* \otimes K_C$. We have $h^0(E) = r$ by Riemann-Roch and the stability of E_0 . The inclusion $\Theta = C \subset \Theta_{E_0}$ means that $h^0(E_0(p)) \geq 1$ for all $p \in C$, or equivalently by Serre duality $h^0(E(-p)) \geq 1$; this implies that the subsheaf F of E generated by the global sections of E is of rank $< r$. Moreover if p does not belong to Δ , it is a smooth point of Θ_{E_0} , and thus satisfies $h^0(E(-p)) = 1$ (see e.g. [L], § V); therefore $\text{rk } F = r - 1$ (otherwise we would have $h^0(E(-p)) \geq h^0(F(-p)) \geq 2$).

(1.3) Let Z be a component of the locus of stable bundles E of rank r and determinant $K_C^{\otimes r}$, with the property that $H^0(E)$ span a sub-bundle of rank $r - 1$ of E . We will prove the inequality $\dim Z \leq \dim |(r - 1)\Theta|$. It implies that the general fibre of $\theta : Z \dashrightarrow \Theta + |(r - 1)\Theta|$ is finite (possibly empty), so the Proposition follows.

Let E be a general element of Z , and let F be the sub-bundle of E spanned by $H^0(E)$. Put $L := \det F$ and $d = \deg F = \deg L$; we have an exact sequence

$$0 \rightarrow L^{-1} \longrightarrow H^0(E) \otimes_{\mathbb{C}} \mathcal{O}_C \longrightarrow F \rightarrow 0 ,$$

hence a linear map $H^0(E)^* \rightarrow H^0(L)$. Let $s = r - \dim H^0(C, F^*)$ be the rank of that map. Then $F = \mathcal{O}_C^{r-s} \oplus G$, where G is a vector bundle of rank $s - 1$ with $h^0(G) = s$, $h^0(G^*) = 0$, which fits into an exact sequence

$$0 \rightarrow L^{-1} \longrightarrow \mathcal{O}_C^s \longrightarrow G \rightarrow 0 .$$

The quotient $\mathcal{M} = E/F$ is the direct sum of a line bundle M and a torsion sheaf \mathcal{T} . We have $c_1(M) + c_1(\mathcal{T}) = rc_1(K_C) - c_1(L)$, and this formula determines M once \mathcal{T} and L are given. We denote by t the length of \mathcal{T} .

(1.4) To summarize, we have associated to a general bundle E in Z integers s, d, t and

- a line bundle L of degree d , and a s -dimensional subspace $V \subset H^0(C, L)$ generating L ; from these data we define G as the cokernel of the natural map $L^{-1} \rightarrow V^* \otimes \mathcal{O}_C$, and put $F := \mathcal{O}_C^{r-s} \oplus G$;

- a torsion sheaf \mathcal{T} of length t and an extension

$$0 \rightarrow F \longrightarrow E \longrightarrow M \oplus \mathcal{T} \rightarrow 0 , \tag{\mathcal{E}}$$

where the line bundle M is determined by $c_1(M) = rc_1(K_C) - c_1(L) - c_1(\mathcal{T})$.

The integers s, d, t are bounded: we have $s \leq r$, $t \leq 2r - d$, and $d < 2(r - 1)$ by the stability of E . Observe also that $d \geq 3$: indeed L is generated by its global sections, and cannot be isomorphic to K_C since otherwise F would contain a copy of K_C , contradicting the stability of E .

The data $(L, V, \mathcal{T}, \mathcal{E})$ are parameterized by a variety dominating Z ; we will bound its dimension. The line bundle L depends on 2 parameters. We have $h^0(L) = d - 1$ since $d \geq 3$, therefore the subspace $V \subset H^0(L)$ depends on $s(d - 1 - s)$ parameters. The torsion sheaf \mathcal{T} depends on t parameters. Over the variety parameterizing these data we build a vector bundle with fibre $\text{Ext}^1(\mathcal{M}, F)$, with $\mathcal{M} = M \oplus \mathcal{T}$, M and F being determined as above. The group $\text{Aut}(\mathcal{M}) \times \text{Aut}(F)$ acts on $\text{Ext}^1(\mathcal{M}, F)$, with the group \mathbb{C}^* of homotheties of \mathcal{M} and F acting in the same way; in fact, since the middle term of the extensions we are interested in is stable, the stabilizer of a general extension class is \mathbb{C}^* . This gives a bound

$$\dim Z \leq 2 + s(d - 1 - s) + t + \dim \text{Ext}^1(\mathcal{M}, F) - \dim \text{Aut}(\mathcal{M}) - \dim \text{Aut}(F) + 1 .$$

Let us estimate the dimensions which appear in the right hand side. We have $\text{Hom}(M, F) \subset \text{Hom}(M, E) = 0$ because E is stable, hence by Riemann-Roch

$$\dim \text{Ext}^1(M \oplus \mathcal{T}, F) = (r - 1)(2r + 1) - dr .$$

The group $\text{Aut}(F) = \text{Aut}(\mathcal{O}_C^{r-s} \oplus G)$ contains the group of matrices $\begin{pmatrix} u & 0 \\ v & \lambda \end{pmatrix}$, with $u \in \text{Aut}(\mathcal{O}_C^{r-s})$, $v \in \text{Hom}(\mathcal{O}_C^{r-s}, G)$, $\lambda \in \mathbb{C}^*$; this group has dimension

$$(r - s)^2 + s(r - s) + 1 = r(r - s) + 1 .$$

The group $\text{Aut}(\mathcal{T})$ has dimension at least t , so similarly $\text{Aut}(\mathcal{M})$ has dimension $\geq 2t + 1$. We get finally:

$$\begin{aligned} \dim Z &\leq 2 + s(d - 1 - s) + t + (r - 1)(2r + 1) - dr - r(r - s) - 2t - 1 \\ &= (r - 1)^2 - 1 - (d - 1 - s)(r - s) - t \end{aligned}$$

Since $d - 1 = h^0(L) \geq s$, this implies $\dim Z \leq (r - 1)^2 - 1 = \dim |(r - 1)\Theta|$ as required. ■

2. Symplectic bundles

Let C be a curve of genus $g \geq 2$, and r a positive integer. The moduli space $\mathcal{SU}_C(r)$ has a natural involution $D : E \mapsto E^*$. Let ι be the involution $L \mapsto K_C \otimes L^{-1}$ of J^{g-1} . The diagram

$$\begin{array}{ccc} \mathcal{SU}_C(r) & \xrightarrow{D} & \mathcal{SU}_C(r) \\ \downarrow \theta & & \downarrow \theta \\ \mathcal{Y} & & \mathcal{Y} \\ |r\Theta| & \xrightarrow{\iota^*} & |r\Theta| \end{array}$$

is commutative.

Assume now that r is even. Let $\mathcal{S}_{\mathcal{P}_C}(r)$ be the moduli space of semi-stable symplectic bundles of rank r on C . This is a normal connected projective variety, with a forgetful morphism to $\mathcal{SU}_C(r)$, which is an embedding on the stable locus. It is contained in the fixed locus of D , thus its image under θ is contained in the fixed locus of ι^* .

This fixed locus is described for instance in [B-L], ch. 4, § 6 (up to a translation from JC to J^{g-1}). The involution ι^* acts linearly on $|r\Theta|$ and has 2 fixed spaces $|r\Theta|^+$ and $|r\Theta|^-$: a symmetric divisor in $|r\Theta|$ is in $|r\Theta|^+$ (resp. $|r\Theta|^-$) if and only if its multiplicity at any theta-characteristic $\kappa \in J^{g-1}$ is even (resp. odd). The dimension of $|r\Theta|^\pm$ is $\frac{1}{2}(r^g \pm 2^g) - 1$.

Proposition 2.1. — $\theta : \mathcal{SU}_C(r) \dashrightarrow |r\Theta|$ induces a rational map from $\mathcal{S}_{\mathcal{P}_C}(r)$ to $|r\Theta|^+$.

Proof: Since $\mathcal{S}_{\mathcal{P}_C}(r)$ is connected, it suffices to find one semi-stable bundle E which admits a symplectic form, and such that $\Theta_E \in |r\Theta|^+$. We take $E = F \oplus F^*$ with the standard alternate form, where $F \in \mathcal{SU}_C(r/2)$ admits a theta divisor. Then $\Theta_E = \Theta_F + \iota^*\Theta_F$. Thus if $\kappa \in J^{g-1}$ is a theta-characteristic, we have $\text{mult}_\kappa(\Theta_E) = 2\text{mult}_\kappa(\Theta_F)$, hence $\Theta_E \in |r\Theta|^+$. ■

Let us go back to the case $g = 2$.

Proposition 2.2. — If C has genus 2, some fibres of $\theta : \mathcal{SU}_C(r) \dashrightarrow |r\Theta|$ have dimension $\geq [\frac{r}{2}] - 1$.

Proof: If r is even, θ induces a rational map $\theta_{sp} : \mathcal{S}_{\mathcal{P}_C}(r) \dashrightarrow |r\Theta|^+$ (Prop. 2.1). We have

$$\dim \mathcal{S}_{\mathcal{P}_C}(r) = \frac{1}{2}r(r+1) \quad , \quad \dim |r\Theta|^+ = \frac{r^2}{2} + 1 \quad ,$$

hence the fibres have dimension $\geq \frac{r}{2} - 1$.

If r is odd, consider the bundle $E \oplus \mathcal{O}_C$, for E general in $\mathcal{S}_{\mathcal{P}_C}(r-1)$; by what we have just seen θ is defined at E , and its fibre at E has dimension $\geq \frac{r-1}{2} - 1$. ■

Remark 2.3. — The degree of $\theta_r : \mathcal{SU}_C(r) \dashrightarrow |r\Theta|$ grows exponentially with r : indeed the commutative diagram

$$\begin{array}{ccc} \mathcal{SU}_C(r) \times \mathcal{SU}_C(s) & \xrightarrow{\oplus} & \mathcal{SU}_C(r+s) \\ \downarrow \theta_r \times \theta_s & & \downarrow \theta_{r+s} \\ |r\Theta| \times |s\Theta| & \xrightarrow{+} & |(r+s)\Theta| \end{array}$$

shows that $\deg \theta_{r+s} \geq \deg \theta_r + \deg \theta_s$. Since $\deg \theta_3 = 2$, we obtain $\deg \theta_r \geq 2^{\lfloor r/3 \rfloor}$ (we expect the actual value to be much higher).

3. Genus 3, rank 3

Recall that if L is a line bundle on C generated by its global sections, the *evaluation bundle* Q_L is defined through the exact sequence

$$0 \rightarrow Q_L^* \longrightarrow H^0(L) \otimes_{\mathbb{C}} \mathcal{O}_C \longrightarrow L \rightarrow 0 ;$$

it has rank $h^0(L) - 1$ and determinant L .

Theorem 3.1.— *Let C be a curve of genus 3, and E_0 a stable vector bundle of rank 3 and degree 0 on C , such that $\Theta_{E_0} \supset \Theta$. Then C is not hyperelliptic, and E_0 is one of the following bundles:*

- a) *The vector bundles $E_N := Q_{K \otimes N} \otimes N^{-1}$, for $N \in J^2 - \Theta$;*
- b) *The vector bundle $\mathcal{E}nd_0(Q_K)$ of traceless endomorphisms of Q_K .*

Conversely, the bundles in a) and b) are stable and admit a theta divisor which contains Θ .

Thus all vector bundles in $\mathcal{SU}_C(3)$ have a theta divisor; in other words, the map $\theta : \mathcal{SU}_C(3) \rightarrow |3\Theta|$ is a morphism. Since $\theta^*\mathcal{O}(1) = \mathcal{L}$ is ample, this morphism is finite: this implies Theorem B of the introduction.

The proof of Theorem 3.1 will occupy the rest of this section. Let E_0 be a stable bundle in $\mathcal{SU}_C(3)$ with $\Theta_{E_0} \supset \Theta$. We will deal mainly with the bundle $E := E^* \otimes K_C$. It has slope 4, degree 12 and satisfies $h^1(E) = h^0(E_0) = 0$ by stability of E_0 , so that $h^0(E) = 6$ by Riemann-Roch. We first establish some properties of E that will be needed later on.

Lemma 3.2.— *Any rank 2 sub-bundle F of E satisfies $h^0(F) \leq 4$.*

Proof: Assume $h^0(F) \geq 5$. Let A be a sub-line bundle of F of maximal degree; this degree is ≥ 2 (since $h^0(F(-p-q)) \geq 1$ for $p, q \in C$) and ≤ 3 by the stability of E . Let $B := F/A$; again by stability of E we have $\deg(F) \leq 7$, hence $\deg(B) \leq 5$. Therefore

$$h^0(F) \leq h^0(A) + h^0(B) \leq 2 + 3 = 5 ;$$

if equality holds, we have $h^0(A) = 2$, $h^0(B) = 3$; moreover the class of the extension

$$0 \rightarrow A \longrightarrow F \longrightarrow B \rightarrow 0$$

must be non-zero (because E cannot contain a line bundle of degree ≥ 4), but must go to zero under the canonical map

$$\mathrm{Ext}^1(B, A) \longrightarrow \mathrm{Hom}(H^0(B), H^1(A)) .$$

In particular this map cannot be injective; equivalently its transpose, the multiplication map

$$H^0(K \otimes A^{-1}) \otimes H^0(B) \longrightarrow H^0(K \otimes A^{-1} \otimes B)$$

cannot be surjective. Now we distinguish two cases:

a) If $\deg(A) = 3$, we must have $A = K_C(-p)$ for some $p \in C$, and $B = K_C$. But then the multiplication map $H^0(\mathcal{O}_C(p)) \otimes H^0(K_C) \xrightarrow{\sim} H^0(K_C(p))$ is an isomorphism.

b) If $\deg(A) = 2$, C is hyperelliptic and A is the hyperelliptic line bundle on C (that is, $h^0(A) = \deg A = 2$). If $B = K_C$, the multiplication map $H^0(A) \otimes H^0(K_C) \rightarrow H^0(A \otimes K_C)$ is surjective. So we must have $\deg(B) = 5$. By the base point free pencil trick, the multiplication map $H^0(A) \otimes H^0(B) \rightarrow H^0(A \otimes B)$ is surjective if and only if $H^1(B \otimes A^{-1}) = 0$, that is, $H^0(K \otimes A \otimes B^{-1}) = 0$. This fails only if $B \cong K(q)$ for some $q \in C$. But in that case B , and therefore also F , are not globally generated. The subsheaf F' of F spanned by $H^0(F)$ has $h^0(F') = 5$, $\deg(F') \leq 6$, and this is impossible by the previous analysis. ■

Lemma 3.3.— *Let p, q be general points of C . Then $h^0(E(-p)) = 3$ and $h^0(E(-p - q)) = 1$.*

Proof: If $h^0(E(-p)) \geq 4$ for all $p \in C$, the global sections of E span a sub-bundle F of rank ≤ 2 with $h^0(F) = 6$. This is impossible by Lemma 3.2. Similarly if $h^0(E(-p - q)) \geq 2$ for all q , the global sections of $E(-p)$ span a sub-line bundle L of $E(-p)$ with $h^0(L) = 3$, hence $\deg L \geq 4$, contradicting the stability of E . ■

Thus the spaces $\mathbb{P}(H^0(E(-p)))$ form a one-dimensional family of planes in $\mathbb{P}(H^0(E)) \cong \mathbb{P}^5$ with the property that any two of them intersect. This situation has been thoroughly analyzed by Morin [M].

Theorem (Morin).— *Any irreducible family of planes in \mathbb{P}^5 such that any two planes of the family intersect is contained in one of the following families:*

- e1) *The planes passing through a given point.*
- e2) *The planes contained in a given hyperplane.*
- e3) *The planes intersecting a given plane along a line.*
- g1) *One of the family of generatrices of a smooth quadric in \mathbb{P}^5 .*
- g2) *The family of planes cutting down a smooth conic on the Veronese surface.*
- g3) *The family of planes in \mathbb{P}^5 tangent to the Veronese surface.*

(3.4) *The elementary cases*

We will first show that our family of planes cannot satisfy one of the elementary conditions e1) to e3).

e1) This would mean that there exists a non-zero section $s \in H^0(E)$ which vanishes at each point of C , a contradiction.

e2) In that case there exists a hyperplane H in $H^0(E)$ such that $H^0(E(-p)) \subset H$ for all p in C . It follows that H span a sub-bundle F of E of rank ≤ 2 , with $h^0(F) \geq 5$; this contradicts Lemma 3.2.

e3) In that case there exists a 3-dimensional subspace W in $H^0(E)$ such that $\dim W \cap H^0(E(-p)) \geq 2$ for all p in C . This implies that W spans a sub-line bundle L of E with $h^0(L) \geq 3$, contradicting the stability of E .

(3.5) *The geometric cases*

Suppose now that our family of planes $\mathbb{P}(H^0(E(-p))) \subset \mathbb{P}(H^0(E))$ is contained in one of the families g1) to g3). We put $V := H^0(E)$ and consider the map $g : C \rightarrow \mathbb{G}(3, V)$ which associates to a general point p of C the subspace $H^0(E(-p))$ of V . This map is defined by the sub-bundle E' of E spanned by $H^0(E)$; that is, the universal exact sequence on $\mathbb{G}(3, V)$

$$0 \rightarrow N \rightarrow V \otimes \mathcal{O}_{\mathbb{G}} \rightarrow Q \rightarrow 0$$

pulls back to the exact sequence

$$0 \rightarrow N_C \rightarrow V \otimes \mathcal{O}_C \rightarrow E' \rightarrow 0$$

on C , where $(N_C)_p = H^0(E(-p))$ for p general in C . The Morin theorem tells us that g factors as

$$g : C \xrightarrow{f} \mathbb{P}^r \hookrightarrow \mathbb{G}(3, V),$$

where $r = 2$ or 3 and \mathbb{P}^r is embedded in $\mathbb{G}(3, V)$ as described in g1) to g3). Conversely if this holds, the vector bundle $E' = g^*Q$ has the property that $h^0(E'(-p-q)) \geq 1$ for all p, q in C .

We will now analyze each of these cases and deduce from this the possibilities for E . We put $L := f^*\mathcal{O}_{\mathbb{P}^r}(1)$.

g1) *Planes in a quadric*

Let U be a 4-dimensional vector space, and $V = \Lambda^2 U$. The equation $v \wedge v = 0$ for $v \in V$ defines a smooth quadric \mathcal{Q} in $\mathbb{P}(V)$. The subvariety of $\mathbb{G}(3, V)$ parameterizing planes contained in \mathcal{Q} has two components, which are exchanged under the automorphism group of \mathcal{Q} . One of these is the image of the map $\mathbb{P}^3 = \mathbb{P}(U^*) \rightarrow \mathbb{G}(3, V)$ which maps the hyperplane $H \subset U$ to the 3-plane $\Lambda^2 H \subset \Lambda^2 U = V$. The Euler exact sequence

$$0 \rightarrow \Omega_{\mathbb{P}^3}^1(1) \rightarrow U \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0$$

gives rise to an exact sequence

$$0 \rightarrow \Lambda^2(\Omega_{\mathbb{P}^3}^1(1)) \longrightarrow \Lambda^2 U \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^3} \longrightarrow \Omega_{\mathbb{P}^3}^1(2) \rightarrow 0$$

which is the pull back to \mathbb{P}^3 of the universal exact sequence on $\mathbb{G}(3, V)$.

Thus $E' \cong f^* \Omega_{\mathbb{P}^3}^1(2)$; the Euler exact sequence twisted by $\mathcal{O}_{\mathbb{P}^3}(1)$ pulls back to

$$0 \rightarrow E' \longrightarrow U \otimes_{\mathbb{C}} L \longrightarrow L^{\otimes 2} \rightarrow 0.$$

This implies $\det E' \cong L^{\otimes 2}$, hence $\deg L \leq 6$. On the other hand the condition $h^0(E'(-p-q)) \geq 1$ for all p, q in \mathbb{C} implies $h^0(L) \geq 3$ and therefore $\deg L \geq 4$. The map $U \rightarrow H^0(L)$ must then be injective, because otherwise a copy of L would inject into E' , contradicting the stability of E . This gives $h^0(L) \geq 4$; the only possibility is $\deg L = 6$ and $h^0(L) = 4$, hence $E' = E$ and $U = H^0(L)$. Thus E is isomorphic to $Q_L^* \otimes L$, where Q_L is the evaluation bundle of L . This vector bundle is analyzed in [B2]: it always admits a theta divisor, and it is stable if and only if C is not hyperelliptic and L is very ample, that is, $L = K_C \otimes N$ with $\deg N = 2$, $h^0(N) = 0$. Dualizing we find $E_0 = E_N := Q_{K \otimes N} \otimes N^{-1}$; this gives case a) of the theorem.

g2) *Secant planes to the Veronese surface*

Let U be a 3-dimensional vector space, and $V = \mathbf{S}^2 U$. The Veronese surface S is the image of the map $u \mapsto u^2$ from $\mathbb{P}(U)$ into $\mathbb{P}(V)$. The family of planes which cut S along a conic is the image of the map $\mathbb{P}^2 = \mathbb{P}(U^*) \rightarrow \mathbb{G}(3, V)$ which maps a 2-plane $H \subset U$ to $\mathbf{S}^2 H \subset \mathbf{S}^2 U$. The pull back to \mathbb{P}^2 of the universal exact sequence on $\mathbb{G}(3, V)$ is the sequence

$$0 \rightarrow \mathbf{S}^2(\Omega_{\mathbb{P}^2}^1(1)) \longrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1)^3 \rightarrow 0$$

obtained by taking the symmetric square of the Euler exact sequence on \mathbb{P}^2 .

Thus E' is isomorphic to $L^{\oplus 3}$. Since E is stable this implies $\deg L \leq 3$, while the inequality $h^0(E'(-p-q)) \geq 1$ imposes $h^0(L) \geq 3$, a contradiction.

g3) *Tangent planes to the Veronese surface*

Consider again the Veronese surface S , image of the square map $\mathbb{P}(U) \rightarrow \mathbb{P}(V)$. The projective tangent bundle of S in $\mathbb{P}(V)$ is $\mathbb{P}_S(\tilde{T}_S)$, where \tilde{T}_S appears in the extension

$$0 \rightarrow \mathcal{O}_S \longrightarrow \tilde{T}_S \longrightarrow T_S \rightarrow 0$$

with class $c_1(\mathcal{O}_{\mathbb{P}(V)}(1)|_S) \in H^1(S, \Omega_S^1)$; the Euler exact sequence provides an isomorphism $\tilde{T}_S \cong U \otimes \mathcal{O}_{\mathbb{P}(U)}(1)$. Similarly we have an extension $\tilde{T}_{\mathbb{P}(V)}$ of $T_{\mathbb{P}(V)}$ by $\mathcal{O}_{\mathbb{P}(V)}$

and an isomorphism $\tilde{T}_{\mathbb{P}(V)} \cong V \otimes \mathcal{O}_{\mathbb{P}(V)}(1)$. These bundles fit into a normal exact sequence

$$0 \rightarrow \tilde{T}_S \longrightarrow \tilde{T}_{\mathbb{P}(V)|S} \longrightarrow N_{S/\mathbb{P}(V)} \rightarrow 0 ,$$

that is, after a twist by $\mathcal{O}_S(-2)$,

$$0 \rightarrow U \otimes \mathcal{O}_S(-1) \longrightarrow V \otimes \mathcal{O}_S \longrightarrow N_{S/\mathbb{P}(V)}(-2) \rightarrow 0 ,$$

which is the pull back to S of the universal exact sequence on $\mathbb{G}(3, V)$. Recall that the second fundamental form gives an isomorphism $N_{S/\mathbb{P}^5} \cong \mathbf{S}^2 T_S$ (see for instance [G-H]).

Thus $E' = \mathbf{S}^2 f^*(T_{\mathbb{P}^2}(-1))$. This gives $\det E' = L^{\otimes 3}$, hence $\deg L \leq 4$. On the other hand we have $h^0(L) \geq 3$: otherwise the image of C in \mathbb{P}^5 is a conic $c \subset S$, and all tangent planes to S along c meet the plane of c along a line, so that we are in case e3). Therefore $L = K_C$, $E = E'$. The Euler exact sequence shows that $f^*(T_{\mathbb{P}^2}(-1))$ is isomorphic to the evaluation bundle Q_K of K_C , so that $E \cong \mathbf{S}^2 Q_K$. Using the canonical isomorphism $\mathbf{S}^2 F \otimes (\det F)^{-1} \xrightarrow{\sim} \mathcal{E}nd_0(F)$ for a rank 2 bundle F we get $E_0 \cong \mathcal{E}nd_0(Q_K)$.

The vector bundles Q_K , and therefore $\mathcal{E}nd_0(Q_K)$, are semi-stable. If C is hyperelliptic, Q_K is isomorphic to $H \oplus H$, where H is the hyperelliptic line bundle, hence $\mathcal{E}nd_0(Q_K) \cong \mathcal{O}_C^{\oplus 3}$.

Assume now that C is not hyperelliptic; then Q_K is stable [P-R]. If $\mathcal{E} := \mathcal{E}nd_0(Q_K)$ is not stable, it admits as sub- or quotient bundle a line bundle of degree 0; this means that there exists a non-zero homomorphism $Q_K \rightarrow Q_K \otimes M$, with $M \in JC$, which must be an isomorphism because Q_K is stable. Taking determinants gives $M^{\otimes 2} \cong \mathcal{O}_C$. Since C is not hyperelliptic M cannot be written $\mathcal{O}_C(p - q)$ with $p, q \in C$; therefore $h^0(Q_K \otimes M) = 0$ [P-R], so that $Q_K \otimes M$ cannot be isomorphic to Q_K .

It remains to prove that \mathcal{E} admits a theta divisor. What we have proved so far is that \mathcal{E} is the only stable rank 3 vector bundle of degree 0 which might possibly satisfy $\Theta_{\mathcal{E}} = J^2$. But if this was the case, all the vector bundles $\mathcal{E} \otimes M$, for $M \in JC$, should have the same property – an obvious contradiction. ■

Remarks 3.6.— a) If we restrict ourselves to $\mathcal{SU}_C(3)$, we find 37 stable bundles, namely $\mathcal{E}nd_0(Q_K)$ and the bundles E_{κ} where κ is an even theta-characteristic. These bundles appear already in [P], in a somewhat disguised form: one can show indeed that E_{κ} is isomorphic to $\mathcal{E}nd_0(A(\kappa, L, x))$, where $A(\kappa, L, x)$ is the *Aronhold bundle* defined in [P] (up to a twist, this bundle depends only on κ).

b) The theta divisor of E_N is determined in [B2]: it is equal to $\Theta + \Delta_N$, where Δ_N is the translate by N of the divisor $C - C$ in JC . The theta divisor

of $\mathcal{E}nd_0(Q_K)$ is $\Theta + \Xi$, where Ξ is an interesting canonical element of $|2\Theta|$. One can show that the trace of Ξ on $\Theta \cong \mathbf{S}^2 C$ is the locus of divisors $p + q$ such that the residual intersection points of C with the line $\langle p, q \rangle$ are harmonically conjugate with respect to p, q (here we view C as a plane quartic).

c) Let $X \subset |3\Theta|$ be the closed subvariety of divisors of the form $\Theta + \Theta_E$ for some E in $\mathcal{SU}_C(2)$. It follows from Theorem 3.1 and the above remarks that the fibre of $\theta : \mathcal{SU}_C(3) \dashrightarrow |3\Theta|$ over a general point of X is reduced to one element, while $\theta^{-1}(\Theta + \Delta_\kappa)$, for κ an even theta-characteristic, has 2 elements, namely E_κ and $\mathcal{O}_C \oplus (Q_K \otimes \kappa^{-1})$. From general principles this implies that the variety $\theta(\mathcal{SU}_C(3))$ is *not normal* at the 36 points $\Theta + \Delta_\kappa$ (see for instance [EGA], 15.5.3).

d) Assume that the Néron-Severi group of JC has rank 1 – this holds if C is general enough. Then a reducible divisor in $|3\Theta|$ must contain a translate of Θ . We thus deduce from Theorem 3.1 that the stable vector bundles of rank 3 and degree 0 on C which admit a reducible theta divisor are those of the form $E_N \otimes M$ or $\mathcal{E}nd_0(Q_K) \otimes M$, for $M \in JC$.

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